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Risk, ambiguity, and the value of diversification

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RISK, AMBIGUITY, AND THE VALUE OF DIVERSIFICATION

Loïc Berger and Louis Eeckhoudt*

Abstract

Diversification is a basic economic principle that helps to hedge against uncertainty. It is therefore intuitive that both risk aversion and ambiguity aversion should positively affect the value of diversification. In this paper, we show that this intuition (1) is true for risk aversion but (2) is not necessarily true for ambiguity aversion. We derive sufficient conditions, showing that, contrary to the economic intuition, ambiguity and ambiguity aversion may actually reduce the diversification value.

Keywords: Diversification, ambiguity aversion, model uncertainty, hedging

JEL Classification: D81

> "... *it is advisable to divide goods which are exposed to some danger into several portions rather than to risk them all together.*" Bernoulli (1738)

1 Introduction

"*Don't put all your eggs in one basket*" is a familiar adage highlighted in many, if not all, textbooks of microeconomics and finance. Put simply, this precept means that an individual should diversify her portfolio among a collection of assets when these are ranked equivalently. The idea behind preferences for diversification comes from basic economic principles. In a world without uncertainty, it simply reflects the desire for variety. In an uncertain world, it reflects the desire to hedge against uncertainty. The intuition suggesting that diversification is valued positively dates back at least to Bernoulli (1738). In his famous article on the measurement of risk, Bernoulli (1738) illustrated the concept using the example of Sempronius who possesses 8000 ducats worth commodities in foreign countries from where they can be transported only by sea. Sempronius asks himself whether to trust equal portions of his commodities to two ships instead

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of one, knowing that each ship may perish during the journey. Under these conditions, Bernoulli (1738, p. 31) maintained that diversification is desirable, by noting "*the value of Sempronius' prospects of success will grow more favorable the smaller the proportion committed to each ship*". Bernoulli further advices "*those who invest their fortune in foreign bills of exchanges and other hazardous enterprises*" to proceed similarly. This paper studies formally the problem posed by Bernoulli in a general uncertain environment, and examines how risk, ambiguity, and preferences affect the value of diversification.

According to Dekel (1989), an important application of the theory of choice under uncertainty is to model markets of uncertain assets. For these markets, preference for diversification is a crucial feature. If the assets are *risky* (i.e., if the true probability law governing the stochastic processes of asset returns is perfectly known), it is well-known that an expected utility (EU) maximizer will exhibit a preference for diversification if and only if she is risk averse (Samuelson, 1967; Rothschild and Stiglitz, 1971). However, as is now widely acknowledged, in most real-life situations assets are uncertain –or *ambiguous*– rather than risky (Uppal and Wang, 2003). As a consequence, agents are uncertain about which probability law (or *model*) to use to describe the asset return processes. How does this additional source of uncertainty affect the basic diversification principle?

In line with the standard assumption of ambiguity aversion (Ellsberg, 1961), economic wisdom suggests the emergence of an hedging demand due to ambiguity. Marinacci (2015, p. 1056), for example, notes that ambiguity aversion, with its desire for robustness, magnifies preference for diversification: "*In general, under ambiguity aversion, hedging against model uncertainty provides a further motif for action diversification, on top of the standard hedging motive against state uncertainty that risk aversion features. [...] Ambiguity aversion thus features a preference for hedging, as first remarked by Schmeidler (1989), who referred to this property as uncertainty aversion.*" Interestingly, this feature does not seem to depend on the particular model of choice under uncertainty taken into consideration. Epstein and Schneider (2010, p. 337), for example, write: "*A theme that is common to smooth models and the multiple prior model is the emergence of hedging demand due to ambiguity.*"

It is therefore surprising that, as we show in this paper, ambiguity and ambiguity aversion do not necessarily increase the value of diversification. In line with Gollier (2011), who shows that the comparative statics of ambiguity aversion on portfolio choices is non-trivial, our findings directly challenge the observational-equivalence result (Anderson et al., 2000; Hansen and Sargent, 2001; Maenhout, 2004) by showing that ambiguity aversion does *not* simply reinforce the effect of risk aversion.[1] As a single counterexample is sufficient to demonstrate our claim that the effect of ambiguity aversion on diversification is not necessarily identical to the effect of risk aversion, we focus on simplified, but not spurious, representations of the uncertain environment. Using these examples, we show that (1) the value of diversification is always positive under risk aversion but, contrary to what suggested by economic wisdom, (2) the presence of ambiguity may *reduce* or *leave unchanged* the individual's desire to diversify, even under (possibly extreme)

[1] As Uppal and Wang (2003) note, the intuition behind the observational-equivalence result is that model uncertainty adds another layer of uncertainty to the riskiness of the consumption process. If the decision maker is ambiguity averse, such that she is pessimistic about alternative models, she will then act as if she is *more risk-averse* in evaluating the given consumption process.

ambiguity aversion. The intuition behind the emergence of such a counterintuitive result can be explained as follows: when Sempronius exhibits preference for using several ships rather than just one, the probability of having at least one accident increases, but the risk is also better spread across a larger number of ships. Under ambiguity, the correct probability model governing the distributions of returns is uncertain. While the diversification principle applies to each model separately, it is also true that when aggregating the different distributions into a single distribution, the positive effect of risk reduction may be reduced.

2 The value of diversification under risk

We consider the problem of Sempronius (or, more generally, of a decision maker, DM), with a sure level of wealth W_0. Sempronius can choose between transporting his commodities of value L (8000 ducats in this case) by a single ship or by n different ships knowing that each ship is independently subject to an identical risk of perishing. For the sake of simplicity, we consider the case examined by Bernoulli (1738) consisting in sending either one or two ships.[2] If the probability distribution of the random event is perfectly known (i.e. if the probability of perishing is p), one faces a simple problem of diversification under *risk*. Under EU, transporting all commodities by a single ship (non-diversification case, ND) yields the value

$$EU^{ND}(W) = pu\left(W_0 - L\right) + (1-p)u\left(W_0\right), \tag{1}$$

where u is the standard von Neumann-Morgenstern (vNM) utility function. Similarly, splitting commodities into two ships (diversification case, D) gives

$$EU^{D}(W) = p^2 u\left(W_0 - L\right) + 2p(1-p)u\left(W_0 - \frac{L}{2}\right) + (1-p)^2 u\left(W_0\right). \tag{2}$$

In these expressions, W is a random variable representing the DM's final wealth. Under ND, it is characterized by the lottery $(W_0 - L, p; W_0, 1-p)$ yielding outcome $W_0 - L$ with probability p, and W_0 otherwise, while under D, it corresponds to the lottery $(W_0 - L, p^2; W_0 - L/2, 2p(1-p);$ $W_0, (1-p)^2)$.

Intuitively, a risk-averse DM will always prefer (2) to (1) because, with two ships, she reduces the risk of loss in the sense of Rothschild and Stiglitz (1970). To show this, we formally define the value of diversification as follows:

Definition 1. The *value of diversification* is the increase in EU resulting from diversification.

Mathematically, it is written as the difference

$$VD = EU^{D}(W) - EU^{ND}(W). \tag{3}$$

Note that this definition is expressed in utility terms, therefore mimicking the definition of *value*

[2] Alternatively, the problem may be considered as one consisting of purchasing a single piece of property of value L, or n multiple pieces of value L/n, knowing that each property is independently subject to identical risk of full loss, or of purchasing shares of securities whose value is described by identical but independent distributions. The generalization of what follows in the case of n ships, properties, or securities, is straightforward.

of information of Marschak and Radner (1972). In line with the early idea of Bernoulli (1738), later discussed in Samuelson (1967) and Rothschild and Stiglitz (1971), we then have

Proposition 1. *The value of diversification is positive if and only if the DM is risk averse:*

$$VD \geq 0 \iff u \text{ is concave.}$$

Proof. All proofs are relegated to the Online Appendix. □

Example 1 To understand these results, consider the risk of transporting the 8000 ducats commodities either by a single ship (risk ND_1) or by two ships (risk D_1). Figure 1 illustrates both risks. For the sake of simplicity, assume that the probability of perishing during the journey is $p = 0.5$. It is then easy to show that the two risks have the same expected value of $\mathbb{E}[ND_1] = \mathbb{E}[D_1] = -4000$ but that D_1 is a *decrease in risk* of ND_1 in the sense of Rothschild and Stiglitz (1970). We can compute the value of diversification in this case as

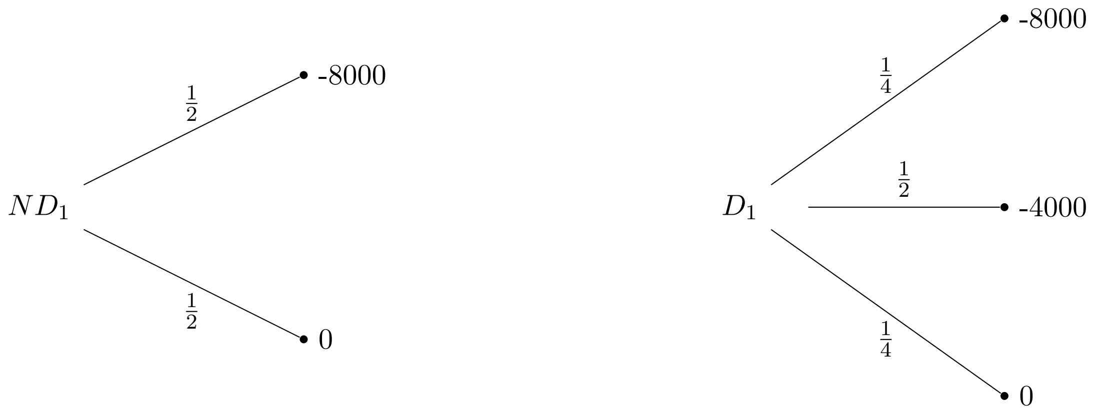

Figure 1: An example of non-diversified (ND_1) and diversified (D_1) risks

$$VD = \frac{1}{4}\left[2u\left(W_0 - 4000\right) - u\left(W_0 - 8000\right) - u\left(W_0\right)\right], \tag{4}$$

which is positive if and only if the function u is concave. The preference for lottery B over lottery A therefore reflects the desire of a risk-averse DM to hedge against the risk of perishing during the journey.

3 The value of diversification under ambiguity

As in most real-life situations however, the exact probability distribution of asset returns might not be perfectly known. In such a situation, the diversification problem is a decision problem under *uncertainty* (or ambiguity) rather than *risk*. Imagine, for example, that the information is imperfect and that Sempronius does not know what the probability p of a perishing is. To help choose between one or two ships, Sempronius relies on the advice of two "experts", indexed by $\theta \in \Theta = \{1, 2\}$. These experts act as advisers by providing their own assessment p_θ of the risk of catastrophe. They refer, for example, to experienced ship-owners, each of whom independently assesses the probability of perishing during the journey.[3] As different probability models exist, the uncertainty surrounding the diversification problem may be decomposed, along the lines of Arrow (1951); Hansen (2014); Marinacci (2015); Hansen and Marinacci (2016), into *model uncertainty* (i.e. what is the correct model, or expert) and *risk* (i.e. what are the potential outcomes, for each model).

To make the diversification decision, the DM can use one of the criteria of decision making under uncertainty developed in the literature (for a review, see Gilboa and Marinacci, 2013). In particular, the DM may either adopt a Bayesian perspective, in which this extra source of uncertainty is quantified by a single probability measure μ, or, because of the incompleteness of information, have multiple priors over the experts' distributions. In what follows, we consider both possibilities in turn. Importantly, note that the uncertainty surrounding the "correctness" of each expert is of a different nature than that encountered in the case of risk. Under risk, the probabilities are well-defined and objectively accepted. For example, the probability of perishing represents the inherent measure of randomness of the catastrophic event. By contrast, the uncertainty about the correctness of the experts is of an epistemic nature such that no *objective* probability measure can be associated with either expert. If the DM still forms a probability measure over the experts' distributions, these probabilities are *subjective* by nature, reflecting the DM's degree of belief in each expert.[4]

Example 2 To illustrate the situations faced by Sempronius, consider the examples presented in Figure 2. In this case, the probability of catastrophe is 3/4 according to expert 1 and 1/4 according to expert 2. If both experts appear equally plausible, Sempronius may attribute a subjective probability 1/2 to each of them.

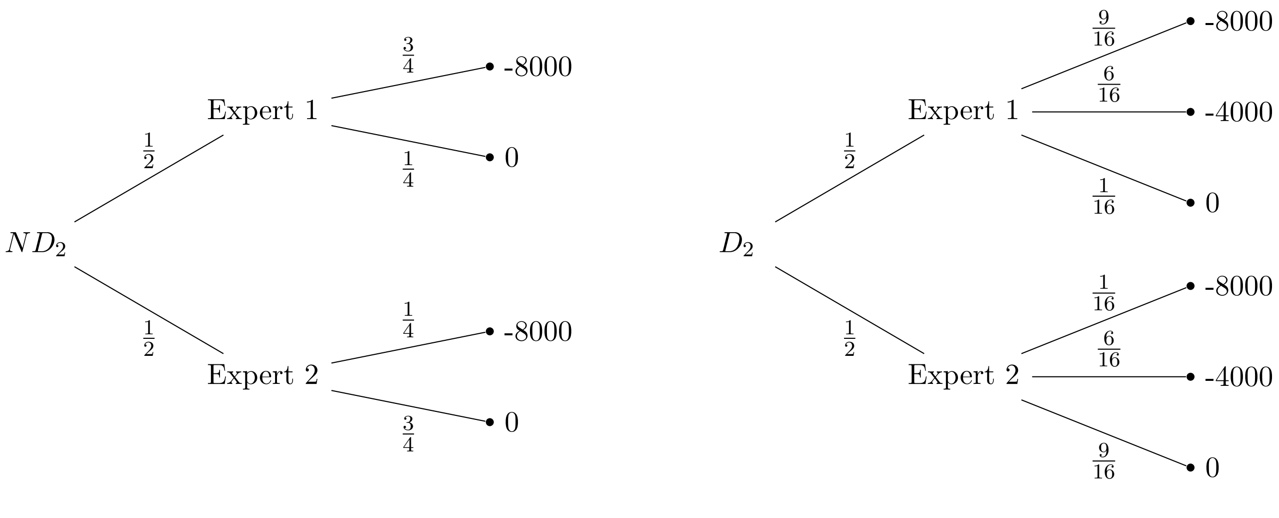

Figure 2: An example of non-diversification (ND_2) and diversification (D_2) under ambiguity

[3]More generally, the experts represent individuals or entities that presumably have more information and/or expertise than the DM and whose information provided is, as such, taken as a datum of the decision problem. This is a standard assumption in classical statistics (Wald, 1950).

[4]These probabilities are, for example, subjectively determined by some higher principles (e.g., the principle of insufficient reason, which may justify symmetric probabilities in the case when no information enables the DM to distinguish the experts).

The literature has shown different ways to deal with this type of ambiguous situation. In what follows, we present some that have recently emerged.

3.1 Subjective expected utility

The subjective expected utility (SEU) is the first criterion we consider. It is a Bayesian criterion original to Savage (1954). In line with Ghirardato and Marinacci (2002), it is used as a benchmark characterizing ambiguity neutrality (AN).[5] The two-stage version of this criterion, which was axiomatized by Cerreia-Vioglio et al. (2013), is particularly useful for the problem we consider. Accordingly, an ambiguity-neutral DM's welfare is evaluated by first computing $EU\left(W(\theta)\right)$ using the probability given by each expert θ separately and then by averaging them using a prior probability measure μ. The utility obtained in the absence of diversification is

$$EU_{AN}^{ND}(W) = \sum_{\theta} \mu\left(\theta\right) \left(EU^{ND}\left(W(\theta)\right)\right) = \sum_{\theta} \mu\left(\theta\right) \left(p_\theta u\left(W_0 - L\right) + (1-p_\theta)u\left(W_0\right)\right), \tag{5}$$

while the utility obtained under diversification is

$$\begin{aligned} EU_{AN}^{D}(W) &= \sum_{\theta} \mu\left(\theta\right) \left(EU^{D}\left(W(\theta)\right)\right) \\ &= \sum_{\theta} \mu\left(\theta\right) \left(p_\theta^2 u\left(W_0 - L\right) + 2p_\theta(1-p_\theta)u\left(W_0 - \frac{L}{2}\right) + (1-p_\theta)^2 u\left(W_0\right)\right). \end{aligned} \tag{6}$$

From these expressions, it should be clear that both risk and model uncertainty are implicitly treated the same way under this criterion. One interpretation may indeed be that the evaluation is performed in two steps: In a first step, the certainty equivalent $u^{-1}\left(EU^j\left(W(\theta)\right)\right)$ of purchasing either one ($j = ND$) or two ($j = D$) properties is computed for each expert θ separately, using the vNM function u representing risk attitude. In a second step, an EU over these certainty equivalents is computed using the same function u together with the prior measure μ. Given the linearity in probabilities, it should be clear that the expressions (5) and (6) can be reduced to take the form of the original SEU representation proposed by Savage (1954), in which $\bar{p} = \sum_\theta \mu(\theta)p_\theta$ is the predictive (subjective) probability that one ship perishes during the journey. Ambiguity neutrality in such a situation therefore results from identical attitudes toward risk and model uncertainty. In line with Proposition 1, the value of diversification under ambiguity neutrality, $VD_{AN} = EU_{AN}^{D}(W) - EU_{AN}^{ND}(W)$, is positive under risk aversion. We can then compare the value of diversification under ambiguity with an hypothetical corresponding risk. Specifically, comparing VD_{AN} and VD_{risk}, in the case $\bar{p} = p$, leads to our first result that challenges the economic intuition.

Proposition 2. *The presence of ambiguity reduces the value of diversification under ambiguity*

[5]The choice of SEU as our benchmark model for AN is a minimal assumption. These preferences intuitively embody AN; while they might not be the only ones (e.g., an alternative definition, proposed by Epstein (1999), relates to probabilistic sophistication), they seem to be the most obvious ones (Gilboa and Marinacci, 2013). This benchmark is moreover particularly well suited for the ambiguity models we consider herein. Specifically, SEU corresponds to a linear ambiguity function ϕ in the smooth ambiguity model of Klibanoff et al. (2005), and Marinacci (2002) shows that it is essentially without loss of generality to assume SEU as the benchmark model for ambiguity neutrality for α-maxmin preferences (for more details, see Jewitt and Mukerji, 2017).

neutrality.

Intuitively, by choosing to transport his commodities by several ships rather than one, Sempronius increases the probability of having at least one accident, but also better spreads the risk across the ships. Under ambiguity, the probability of each ship perishing during the journey is unknown. While the diversification principle applies to each probability model given by the experts separately, the positive effect of risk reduction is reduced when averaging the different distributions using the classical SEU, such that $VD_{AN} < VD_{risk}$. More technically, this can be explained by the concavity of the value of diversification function in p under risk (see Online Appendix, equation (A.1)). As the presence of ambiguity may be seen as a mean-preserving spread of the loss probability, the value of diversification is reduced when the probability becomes unknown. Remark however that, in accordance with Proposition 1, $0 \leq VD_{AN}$ as long as the DM is risk averse.

Example 3 To understand this result, consider Figure 3, which presents the reduced forms of the ambiguous situations of Figure 2, in the case of ambiguity neutrality. The value of

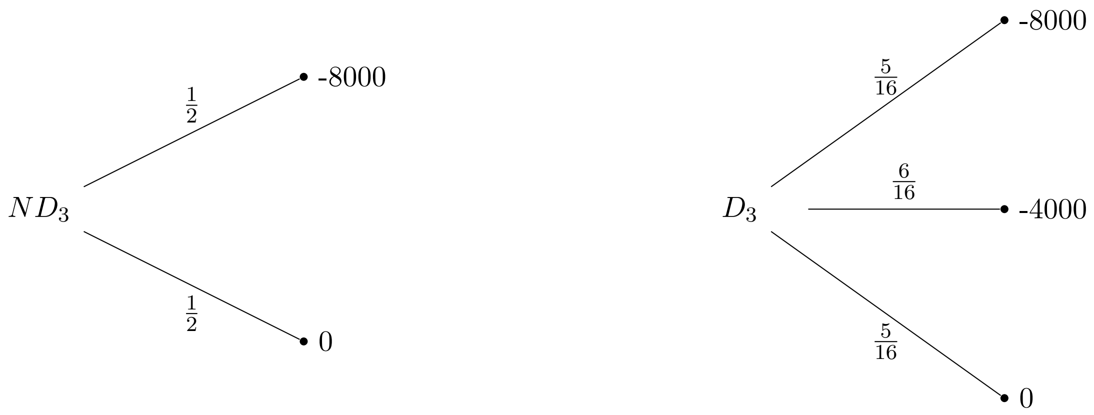

Figure 3: An example of non-diversification (ND_3) and diversification (D_3) under ambiguity neutrality

diversification in this situation is

$$VD_{AN} = \frac{3}{16}\left[2u\left(W_0 - 4000\right) - u\left(W_0 - 8000\right) - u\left(W_0\right)\right], \tag{7}$$

which is positive under risk aversion, but lower than VD_{risk} computed in (4). The presence of ambiguity therefore *reduces* the desire to hedge against independent potential losses under ambiguity neutrality.

It is then important to understand what happens for ambiguity-non-neutral DMs and, in particular, whether ambiguity aversion magnifies the preference for diversification, as the economic intuition would suggest, or not. Given the broad evidence that most individuals treat subjective probabilities differently than objective ones (Ellsberg, 1961), several lines of research have appeared, leading to the emergence of various decision criteria allowing for ambiguity aversion.

3.2 α-maxmin preferences

One departure from the SEU framework originates from the work of Gilboa and Schmeidler (1989). Their *multiple priors* approach relaxes the assumption of model uncertainty being quantified by a single probability measure μ and instead allows for the possibility of multiple priors belonging to a set C.[6] Under the α-maxmin criterion of Ghirardato et al. (2004), both the least favorable among all the classical subjective expected utilities determined by each prior μ in C and the most favorable one appear respectively with weights α and $1-\alpha$. The multiple priors maxmin model of Gilboa and Schmeidler (1989) naturally emerges as a special case when $\alpha = 1$, while the classical SEU criterion is recovered when the set C contains only one element. When C consists of all possible prior probability measures, we recover the criterion due to Hurwicz when $\alpha \in (0,1)$ and to Wald when $\alpha = 1$. In what follows, these are the versions of the α-maxmin model we use to investigate our diversification problem under ambiguity.

The utility of sending either one or two ships under the α-maxmin model is

$$EU^j_{\alpha-mxm}(W) = \alpha \min_\theta EU^j\left(W(\theta)\right) + (1-\alpha)\max_\theta EU^j\left(W(\theta)\right), \tag{8}$$

where $j \in \{D, ND\}$. In this expression, α can be interpreted as a measure of the intensity of ambiguity aversion: going from 0 when the DM is extremely optimistic (and considers only the expert giving the lowest probability of accident) to 1 when the DM is extremely pessimistic (and considers only the highest probability of accident). We can then show that the value of diversification, $VD_{\alpha-mxm} = EU^D_{\alpha-mxm}(W) - EU^{ND}_{\alpha-mxm}(W)$, may be higher than, lower than, or equal to VD_{AN}. In other words, ambiguity aversion has an undetermined effect on the value of diversification. This result challenges economic wisdom, according to which ambiguity aversion magnifies the preference for diversification. In what follows, we present sufficient conditions for ambiguity aversion to decrease the value of diversification.

Proposition 3.1. *Under the α–maxmin model, ambiguity aversion decreases the value of diversification if $p_\theta \geq 0.5$ for all θ.*

Intuitively, when there are only two experts, the condition $\alpha > \mu$ may be interpreted as ambiguity aversion: A higher weight than the subjective belief the DM has over the more pessimistic expert is given to this expert. Yet ambiguity aversion can decrease the value of diversification if the probabilities given by the experts are both higher than 0.5. In this case indeed, the value of diversification obtained following the more pessimistic expert is necessarily lower than the one obtained following the more optimistic one. As the value of diversification, under the α-maxmin criterion, may be computed as an α-average of the values of diversifications computed from each expert separately (see Online Appendix, equation (A.3)), ambiguity aversion leads to a reduction of the value of diversification.

Alternatively, the value of diversification is the same as under ambiguity neutrality in the following situation.

[6] This set of possible priors C incorporates both the attitude toward ambiguity and an information component: a smaller set C reflecting, for example, both better information and/or less ambiguity aversion.

Proposition 3.2. *Under the α–maxmin model, the value of diversification is independent of the intensity of ambiguity aversion if $p_1 = 1 - p_2$.*

In other words, $dVD_{\alpha-mxm}/d\alpha = 0$ when $p_1 = 1 - p_2$, such that whether the DM is extremely ambiguity averse ($\alpha = 1$) or extremely ambiguity loving ($\alpha = 0$) does not influence the diversification decision, which depends on risk attitude only. Intuitively, it can be explained by the fact that the value of diversification following each expert separately is exactly the same in this case. (Note that the value of diversification is also the same as under ambiguity neutrality.)

Example 4 To understand what precedes, consider again the ambiguous situations ND_2 and D_2 presented in Figure 2. If Sempronius is extremely ambiguity averse, he considers expert 1 only and compares the two situations illustrated in Figure 4. In this case, the value of diversification

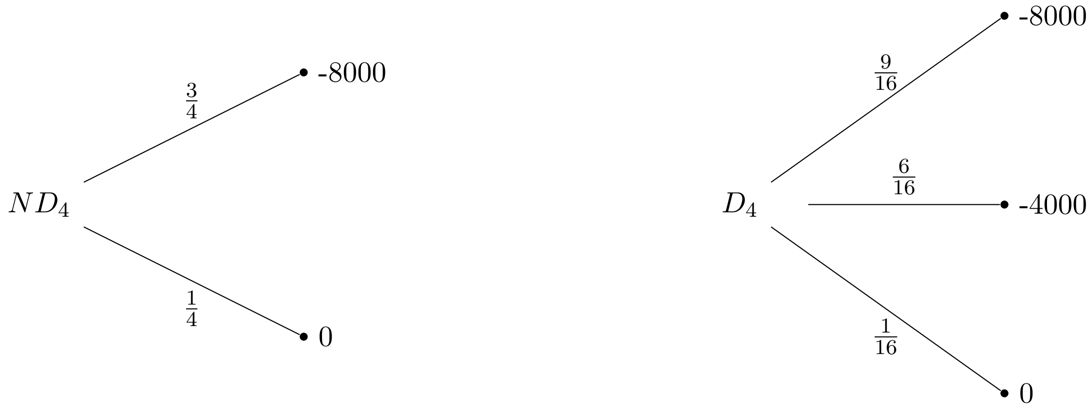

Figure 4: An example of non-diversification (ND_4) and diversification (D_4) under extreme ambiguity aversion ($\alpha = 1$)

is exactly the same as under ambiguity neutrality:

$$VD_{\alpha-mxm} = \frac{3}{16}\left[2u\left(W_0 - 500\right) - u\left(W_0 - 1000\right) - u\left(W_0\right)\right]. \tag{9}$$

In line with what precedes, it is moreover positive under risk aversion.

3.3 Smooth ambiguity preferences

The smooth model of ambiguity, proposed by Klibanoff et al. (2005), makes a distinction between risk and model uncertainty (see also Marinacci, 2015). This model has been widely used, both because of the flexibility it offers and because the function representing the DM's preference for uncertainty is smooth and thus differentiable. The smooth model is built within a Bayesian framework, in which both layers of uncertainty are described by standard probability measures. Ambiguity aversion is incorporated by means of a concave function ϕ, which can be written as $\phi \equiv v \circ u^{-1}$, such that it results from a stronger aversion to model uncertainty (represented by function v) than to risk (represented by the vNM function u). Keeping the same notations as before, the utility of purchasing one or two distinct properties is

$$EU^j_{smt}(W) \quad = \quad \phi^{-1}\left(\sum_{\theta} \mu\left(\theta\right)\phi\left(EU^j\left(W(\theta)\right)\right)\right). \tag{10}$$

In this case, we obtain the utility by first computing $EU^j\left(W(\theta)\right)$ for each expert θ separately and then aggregating them by means of a certainty equivalent using the ambiguity aversion function ϕ and the prior measure μ. (Note that the utility in (10) is expressed in the same units as encountered before.) The value of diversification in the smooth model, $VD_{smt} = EU^D_{smt}(W) - EU^{ND}_{smt}(W)$, therefore depends on the attitude toward both risk and ambiguity, represented by the concavity of functions u and ϕ, respectively. Specifically, the value of diversification will be positive if the DM is both risk averse and ambiguity averse. To show this, we write the value of diversification as

$$VD_{smt} = \phi^{-1}\left(\sum_{\theta} \mu\left(\theta\right)\phi\left(EU^{ND}\left(W(\theta)\right) + \Delta(\theta)\right)\right) - \phi^{-1}\left(\sum_{\theta} \mu\left(\theta\right)\phi\left(EU^{ND}\left(W(\theta)\right)\right)\right), \tag{11}$$

such that, for each expert, the difference in the EU under diversification is larger than that under no-diversification, given that $\Delta(\theta) \geq 0,$ for all $\theta \in \Theta$ under risk aversion (Proposition 1). Yet, when we compare VD_{smt} and VD_{AN}, the intuition that ambiguity aversion provides an extra motive for diversification is seriously challenged. This is not the case, for example, in the following situations.

Proposition 4.1. *Under the smooth ambiguity model, ambiguity aversion decreases the value of diversification if one of the following conditions is satisfied:*

(i) The DM exhibits increasing absolute ambiguity aversion (IAAA) and $p_1 = 1 - p_2$, or

(ii) The DM exhibits constant or increasing absolute ambiguity aversion and $p_\theta \geq 0.5$ for all θ.

Proposition 4.2. *Under the smooth ambiguity model, the value of diversification is independent of the intensity of ambiguity aversion if the DM exhibits constant absolute ambiguity aversion (CAAA) and $p_1 = 1 - p_2$.*

Example 5 To understand these results, consider again the examples provided in Figure 2. In this case, the values of diversification of each expert separately are exactly the same:

$$\Delta\left(\theta\right) \equiv EU^D\left(W\left(\theta\right)\right) - EU^{ND}\left(W\left(\theta\right)\right) = \Delta \quad \text{for all } \theta. \tag{12}$$

In addition, they are equal to VD_{AN} computed in expression (7). *Ceteris paribus*, $EU^D_{smt}(W)$ is thus simply computed at a higher level of EU than $EU^{ND}_{smt}(W)$. For a risk-averse DM, the value of diversification under ambiguity aversion is therefore higher than that under ambiguity neutrality if and only if the DM exhibits decreasing absolute ambiguity aversion (DAAA). By contrast, VD_{smt} is lower than VD_{AN} if the risk-averse DM exhibits IAAA, and $VD_{smt} = VD_{AN}$

in the case of CAAA.[7] Specifically, if the ambiguity aversion function is of the exponential type, e.g. $\phi(x) = -e^{-\lambda x}$, the coefficient of absolute ambiguity aversion (λ) is constant, and the value of diversification is

$$VD_{smt} = \frac{3}{16}\left[2u\left(W_0 - 500\right) - u\left(W_0 - 1000\right) - u\left(W_0\right)\right], \tag{13}$$

as under AN.

4 Concluding remarks and related literature

Modeling markets of uncertain assets is an important application of the theory of choice under uncertainty. For these markets, preference for diversification is a crucial feature. In this paper, we proved that (1) the value of diversification is positive under risk aversion and, contrary to the economic intuition, (2) ambiguity aversion does *not* necessarily magnify the preference for hedging that risk aversion features. Using simple counterexamples, we showed that the presence of ambiguity may instead *reduce* the DM's desire to diversify, even under ambiguity aversion.

Related Literature Our study is related to different strands of literature. First, it sheds some light on a recent debate that took place between Epstein (2010) and Klibanoff et al. (2012). One of the key issues discussed was whether the smooth ambiguity model of Klibanoff et al. (2005) accommodates preferences for diversification. (Note, however, that in that case, *diversification* amounts to *randomization*, and concerns whether, when faced with two ambiguous securities, a DM prefers randomizing 50-50 over the two securities or putting all her wealth into one or the other asset, see also Eichberger et al., 1996; Klibanoff, 2001.) Whereas Epstein (2010) argued that a preference for diversification was not intuitive and that the smooth model was unable to accommodate standard preferences under ambiguity, Klibanoff et al. (2012) maintained that a preference for diversifying across sources of uncertainty is natural and captured by the smooth model.

To understand the distinction between the two views, and make the link with our results, consider the second thought experiment proposed by Epstein (2010, Section 3). In this setup, there are two urns, each containing 50 balls of either the color red (R) or the color blue (B). The proportions of red and blue balls in each urn are generated independently. One ball is drawn from each urn, and the DM considers bets on the colors of the balls with different outcomes as presented in Table 1, where $c^* > c$ and $(c^*, \frac{1}{2}; c, \frac{1}{2})$ represents the 50-50 lottery over outcomes c^* and c. While symmetry suggests indifference between f_1 (i.e. betting on first urn) and f_2 (i.e. betting on the second urn), Epstein (2010) argued that the two urns do not hedge each other, so that it is natural for a strictly ambiguity-averse DM to exhibit $f_3 \sim f_1 \sim f_2$. This ranking of preferences is however incompatible with any concave ambiguity function ϕ in the

[7] Building on Pratt (1964), Klibanoff et al. (2005) call the ratio $\lambda(x) = -\phi''(x)/\phi'(x)$ the coefficient of absolute ambiguity aversion at the level of EU x. DAAA (resp. CAAA, IAAA) therefore means that $\partial\lambda(x)/\partial x < 0$ (resp. $= 0, > 0$). Note that DAAA is, in some situations, a sufficient condition for ambiguity prudence (i.e. whether a DM is willing to save more because of the presence of ambiguity; see Berger, 2014). DAAA is also supported empirically by Berger and Bosetti (2020).

Table 1: Acts with Monetary Payoffs for Experiment 2

	Bets for Experiment 2			
	R_1R_2	R_1B_2	B_1R_2	B_1B_2
f_1	c^*	c^*	c	c
f_2	c^*	c	c^*	c
f_3	c^*	$(c^*, \frac{1}{2}; c, \frac{1}{2})$	$(c^*, \frac{1}{2}; c, \frac{1}{2})$	c
f_4	c^*	$\frac{1}{2}c^* + \frac{1}{2}c$	$\frac{1}{2}c^* + \frac{1}{2}c$	c

Notes: R_1B_2 is the event that a red ball is drawn from the first urn and a blue ball is drawn from the second urn, and so forth.

smooth model. By contrast, Klibanoff et al. (2012) argued that there is a good reason to expect $f_3 \succ f_1 \sim f_2$, as the evaluation of f_3 depends on the compositions of both urns, in that the DM's exposure across the two urns is diversified (which is naturally valuable under ambiguity aversion). Importantly, note that the notion of diversification across independent sources of ambiguity considered by Epstein (2010) and Klibanoff et al. (2012) is different from the more fundamental notion of diversification across bets on independent risks that we consider herein (in line with Samuelson, 1967). To understand this, consider act f_4. We can interpret this as the result of one bet on each urn, each giving $\frac{1}{2}c^*$ if the color of the drawn ball is red and $\frac{1}{2}c$ if it is blue. While it might indeed seem naturally valuable for an ambiguity-averse DM to hedge against uncertainty by splitting the bet into two parts as in f_4, what we showed in this paper is that this is not necessarily the case. In particular, our results reveal that the pattern $f_4 \sim f_1 \sim f_2$ is not incompatible with ambiguity aversion. Specifically, these preferences are obtained when the DM is risk neutral (see our Example 5). Moreover, as acts f_3 and f_4 are considered equivalently under risk neutrality, our results lend support to the idea that ambiguity aversion is compatible with $f_3 \sim f_4 \sim f_1 \sim f_2$. Yet we agree with Klibanoff et al. (2012) that, in general, $f_4 \succ f_3 \succ f_1 \sim f_2$ when the DM is both risk and ambiguity averse, such that the value of diversification is positive (in line with our Proposition 1).

Second, while our work follows a simple economic-oriented approach based on preferences maximization, it enables us to reach the same conclusions as a strand of literature that overlaps between economics and finance. For example, Mukerji and Tallon (2001) argued that ambiguity-averse DMs may not want to diversify because diversification would expose them to too many sources of ambiguity. More generally, this literature suggests that ambiguity aversion can provide an explanation for the portfolio under-diversification puzzle. Such results were recently obtained by Boyle et al. (2012), who used a theoretical static model of portfolio choice to incorporate ambiguity about the true distribution of each asset's return. Similar results also appear in Guidolin and Liu (2016), whose approach takes into account ambiguity aversion over a capital asset pricing model to show that an ambiguity-averse investor may rationally hold a substantially under-diversified portfolio relative to the standard mean-variance portfolio. Finally, from an empirical perspective, note that this relationship between ambiguity aversion and under-diversification was observed by Dimmock et al. (2016) and Bianchi and Tallon (2018).

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Online Appendix

Proofs of Propositions 1 and 2. From expression (1) and (2), we can easily compute the value of diversification under risk as

$$VD_{risk} = p(1-p)\left[2u\left(W_0 - \frac{L}{2}\right) - u\left(W_0 - L\right) - u\left(W_0\right)\right], \tag{A.1}$$

which is positive if and only if u is concave.

Similarly, using expressions (5) and (6), we can compute the value of diversification under ambiguity neutrality as

$$VD_{AN} = \left[\mu p_1(1-p_1) + (1-\mu)\,p_2(1-p_2)\right]\left[2u\left(W_0 - \frac{L}{2}\right) - u\left(W_0 - L\right) - u\left(W_0\right)\right], \tag{A.2}$$

if we let $\mu(1) \equiv \mu$ be the subjective weight attached to the first expert and $\mu(2) \equiv 1-\mu$ the weight attached to the second expert. Comparison of expressions (A.1) and (A.2) leads to the result if and only if

$$\mu p_1(1-p_1) + (1-\mu)\,p_2(1-p_2) \leq p(1-p).$$

It is then easy to show that this is always the case, because $p = \mu p_1 + (1-\mu)p_2$. □

Proofs of Propositions 3.1 and 3.2. Without loss of generality, let us assume $p_1 > p_2$. Then, the value of diversification under the $\alpha - maxmin$ criterion is

$$VD_{\alpha - mxm} = \left[\alpha p_1\left(1-p_1\right) + (1-\alpha)\,p_2(1-p_2)\right]\left[2u\left(W_0 - \frac{L}{2}\right) - u\left(W_0 - L\right) - u\left(W_0\right)\right]. \tag{A.3}$$

Comparing expressions (A.2) and (A.3), we can then show that

$$VD_{\alpha - mxm} \leq VD_{AN} \iff 0 \leq (\mu - \alpha)\left[p_1\left(1-p_1\right) - p_2\left(1-p_2\right)\right].$$

Ambiguity aversion implies that $\alpha > \mu$. It then follows that $VD_{\alpha-mxm} < VD_{AN}$, when $p_1(1-p_1) < p_2(1-p_2)$, and that $VD_{\alpha-mxm} = VD_{AN}$, when $p_1 = 1-p_2$. □

Proofs of Propositions 4.1 and 4.2. First, note that the value of diversification under the smooth model in (11) is lower than the one under ambiguity neutrality if

$$\phi^{-1}\left(\sum_{\theta}\mu\left(\theta\right)\phi\left(EU^{ND}\left(W(\theta)\right) + \Delta(\theta)\right)\right) \leq \phi^{-1}\left(\sum_{\theta}\mu\left(\theta\right)\phi\left(EU^{ND}\left(W(\theta)\right)\right)\right) + \sum_{\theta}\mu\left(\theta\right)\Delta\left(\theta\right). \tag{A.4}$$

In this expression, we can interpret $\Delta(\theta) \equiv p_\theta(1-p_\theta)\left[2u\left(W_0 - \frac{L}{2}\right) - u\left(W_0 - L\right) - u\left(W_0\right)\right]$ as the value of diversification according to expert θ. The following lemma is useful to pursue.

Lemma 1. *Decreasing (resp. constant, and increasing) absolute risk aversion is equivalent to*

$$u^{-1}\left(\mathbb{E}\left[u\left(\tilde{x}+z\right)\right]\right) > (=,<)\ u^{-1}\left(\mathbb{E}\left[u\left(\tilde{x}\right)\right]\right) + z, \tag{A.5}$$

where $\mathbb{E}$ is the expectation operator taken over the random variable $\tilde{x}$ and $z \in \mathbb{R}_{>0}$.

We can then make use of Lemma 1 with the ambiguity function ϕ and note that under IAAA (resp. CAAA), we have

$$\mathbb{E}_\Delta\left[\phi^{-1}\left(\mathbb{E}_\theta\left[\phi\left(EU^{ND}(W(\tilde{\theta})) + \tilde{\Delta}\right)\right]\right)\right] < (=)\, \phi^{-1}\left(\mathbb{E}_\theta\left[\phi\left(EU^{ND}(W(\tilde{\theta}))\right)\right]\right) + \mathbb{E}_\Delta\left[\tilde{\Delta}\right]. \tag{A.6}$$

Indeed, if inequality (A.5) holds for all non-negative Δ, it also holds on average. Using Jensen's inequality, we also know that, under the strict concavity of ϕ (convexity of ϕ^{-1}):

$$\phi^{-1}\left(\mathbb{E}_\Delta\left[\mathbb{E}_\theta\left[\phi\left(EU^{ND}(W(\tilde{\theta})) + \tilde{\Delta}\right)\right]\right]\right) < \mathbb{E}_\Delta\left[\phi^{-1}\left(\mathbb{E}_\theta\left[\phi\left(EU^{ND}(W(\tilde{\theta})) + \tilde{\Delta}\right)\right]\right)\right]. \tag{A.7}$$

What remains to be shown is that

$$\phi^{-1}\left(\sum_\theta \mu\left(\theta\right)\phi\left(EU^{ND}\left(W(\theta)\right) + \Delta(\theta)\right)\right) < \phi^{-1}\left(\mathbb{E}_\Delta\left[\mathbb{E}_\theta\left[\phi\left(EU^{ND}(W(\tilde{\theta})) + \tilde{\Delta}\right)\right]\right]\right). \tag{A.8}$$

Without loss of generality, let us assume that $p_1 > p_2$. It is easy to show that, under the concavity of ϕ, inequality (A.8) is satisfied if $\Delta(2) > \Delta(1)$, for which a sufficient condition is $p_1 > p_2 \geq 0.5$. Combining (A.6), (A.7) and (A.8) gives the result.

Finally, if $p_1 = 1 - p_2$, the value of diversification, for each expert, is constant, $\Delta(\theta) = \Delta$ for all $\theta \in \Theta$, and the result trivially follows from (A.5).

□